Doubling by Reversing

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A while ago, it occurred to me that reversing the digits of the number 37 gave a number that was almost double its value:

$$73 = 37 \times 2 - 1$$
$$\frac{73}{37} = 1.\overline{972}$$

I began to wonder if a integer $n > 9$ existed such that if m is created by reversing the digits of n, $m = 2n$.

As is often the case, my first approach was based in analysis. I looked at the two-digit case by examining one equation in two unknown integers $0 \leq a, b \leq 9$:

$$m = 10a + b, \quad n = 10b + a$$
$$\frac{m}{n} = \frac{10a + b}{10b + a} = 2$$
$$8a = 19b$$

Clearly this equation has no solutions with the given restrictions on a and b. This was sufficient to show that no two-digit numbers have the desired property.

The three-digit case is more complicated but works similarly:

$$m = 100a + 10b + c, \quad n = 100c + 10b + a$$
$$\frac{m}{n} = \frac{100a + 10b + c}{100c + 10b + a} = 2$$
$$98a - 10b = 199c$$

The only solution to this equation with $0 \leq c < a \leq 9$ is:

$$a = 6, \quad b = 19, \quad c = 2$$

Thus there are no three-digit numbers with the desired property either.

Given a fixed number of digits d, we could use this analytical approach to create an equation with d unknowns and examine a finite number of solutions to that equation. However, it doesn't give us a general rule for all numbers, regardless of how many digits they have. We need a different approach—and in this case, we turn to simple arithmetic.

For ease, we denote the first (greatest-value) digit of a positive integer a as a_{first}. Similarly, we denote the last (least-value) digit of a as a_{last}. We also define $\text{rev}(a_b)$ as the positive integer c_b whose digits are the reverse of the digits of a_b, ignoring leading zeroes. (When b is omitted, we assume base 10.)

Fix an arbitrary integer $n > 9$. Suppose $m = 2n = \text{rev}(n)$; we examine the properties of the first and last digits of m and n to see if such an m can exist.

Since $m = 2n$, m is even, so m_{last} must be even, and n_{first} is also even. Further, $n_{\text{first}} < 5$; otherwise $2n$ would have an extra digit. Thus n_{first} is either 2 or 4. m_{last} cannot be 0 because $\text{rev}(n)$ ignores leading zeroes in n, so n_{last} cannot be 0 or 5.
Suppose $n_{\text{first}} = 2$; then $m_{\text{last}} = 2$, and m_{first} is either 4 or 5, depending on if doubling n caused a carry. It cannot be 5, since n_{last} would then be 5, so it must be 4. But then $n_{\text{last}} = 4$, and $(2n)_{\text{last}} = m_{\text{last}} = 8$—a contradiction. Alternatively, if $n_{\text{first}} = 4$, then $m_{\text{last}} = 4$, and m_{first} is either 8 or 9, depending on if doubling n caused a carry. Then n_{last} must be 8 or 9. But if $n_{\text{last}} = 8$, $(2n)_{\text{last}} = m_{\text{last}} = 6$, and if $n_{\text{last}} = 9$, $(2n)_{\text{last}} = m_{\text{last}} = 8$—another contradiction.
Therefore, no m exists such that $m = 2n = \text{rev}(n)$.

While no base-10 integer yields its exact double when its digits are reversed, many numbers come relatively close. 73/37, the first example, is off by only 1.35%. The following table contains the twenty positive integers less than 100000000 that come (relatively) closest to doubling.

n	$\frac{\text{rev}(n)}{n}$	%Error
39999997	$2 - \frac{1}{39999997}$	0.00000125000009375
29999995	$2 + \frac{2}{29999995}$	0.0000033333338888898
49999999	$2 - \frac{4}{49999999}$	0.00000400000008
46000029	$2 + \frac{6}{46000029}$	0.00000652173501890618
42999958	$2 + \frac{4}{21499979}$	0.00000930233466739665
36000027	$2 + \frac{1}{4000003}$	0.00001249999062500703
19999993	$2 + \frac{5}{19999993}$	0.00001250000437500153
3999997	$2 - \frac{1}{39999997}$	0.00001250000937500703
10000002	$2 - \frac{1}{3333334}$	0.00001499999700000059
20000004	$2 - \frac{1}{3333334}$	0.00001499999700000059
30000006	$2 - \frac{1}{3333334}$	0.00001499999700000059
40000008	$2 - \frac{1}{3333334}$	0.00001499999700000059
32999956	$2 + \frac{1}{2999996}$	0.00001666668888891851
26000025	$2 + \frac{4}{8666675}$	0.0000230769008875953
43999978	$2 - \frac{1}{1999999}$	0.00002500001250000625
47000049	$2 - \frac{8}{15666683}$	0.00002553188827526541
48999979	$2 + \frac{26}{48999979}$	0.00002653062361516522
33999976	$2 - \frac{19}{33999976}$	0.00002794119619378554
37000047	$2 - \frac{1}{1761907}$	0.00002837834233021379
22999954	$2 + \frac{7}{11499977}$	0.0000304348434783826

These numbers have some common properties. Each consists of two outer blocks of digits and an inner block containing either all 0s or all 9s. Starting from the outside in, each digit of the first outer block is doubled in the second outer block, sometimes with 1 added to it. The inner block's contents are determined by the need to carry: if the first digit of the second outer block is odd, a carry is present when doubling, and the inner block contains 9s to propagate the carry. Otherwise the inner block contains 0s. When more digits are present, these patterns may repeat themselves, but each iteration contains the same basic structure.

While the preceding numbers are structured similarly, there isn't a trivial pattern to the digits themselves. In contrast, a strong pattern appears when considering the d-digit integer that comes closest to doubling for each d:

d	n
2	37
3	397
4	3997
5	39997
6	399997
7	3999997
8	39999997

These "best" numbers are of the form:

$$n_d = 4 \times 10^{d-1} - 3, \quad d > 1$$

Reversing them creates:

$$\mathrm{rev}(n_d) = 8 \times 10^{d-1} - 7$$

Note that $2n_d - \mathrm{rev}(n_d) = 1$, regardless of the value of d. This means:

$$\lim_{d\to\infty} 2 - \frac{\mathrm{rev}(n_d)}{n_d} = \lim_{d\to\infty} \frac{1}{n_d} = 0$$

From this we see that for any real $\varepsilon > 0$, there exists an integer n' such that $\left|2 - \frac{\mathrm{rev}(n')}{n'}\right| < \varepsilon$. In other words, while no integer can be exactly doubled by reversing its digits, we can find an integer that comes arbitrarily close to being doubled.

Can we find numbers that, when reversed, exactly reach other multiples—disregarding trivial cases such as palindromic numbers? It should be noted that in base 10, we can only consider multiples up to 9; multiplying by 10 or larger would increase the number of digits in the result. Indeed there are two multiples for which there exist numbers that reverse:

n	$\frac{\mathrm{rev}(n)}{n}$
2178	4
1089	9

Note that the two working multiples are $9 = \frac{10}{1} - 1$ and $4 = \frac{10}{2} - 1$. We will return to this point later.

So far, we've considered numbers in base 10, but choosing a different base will affect the value of the number produced when reversing. For example, $37 = 25_{16}$, but $52_{16} = 82 \neq 73$. Can different bases yield numbers that exactly double when their digits are reversed?

The answer is yes. For bases $3 \leq b \leq 16$, the following table shows the smallest numbers n_b that either exactly double when reversed or are off by $\frac{1}{n_b}$. With the exception of 2, which adds one digit to a number when it doubles, each base not of the form $b = 3a + 1$, $a \in \mathbb{N}$ has at least one number that exactly doubles when its digits are reversed. Perhaps surprisingly, base 10 is actually in the minority!

b	$n_b\,(n)$	$\frac{\mathrm{rev}(n_b)}{n_b}$
3	$1012_3\,(32)$	2
4	$13_4\,(7)$	$2-\frac{1}{7}$
5	$13_5\,(8)$	2
6	$2134_6\,(490)$	2
7	$25_7\,(19)$	$2-\frac{1}{19}$
8	$25_8\,(21)$	2
9	$3256_9\,(2400)$	2
10	37	$2-\frac{1}{37}$
11	$37_{11}\,(40)$	2
12	$4378_{12}\,(7436)$	2
13	$49_{13}\,(61)$	$2-\frac{1}{61}$
14	$49_{14}\,(65)$	2
15	$549\mathrm{A}_{15}\,(17920)$	2
16	$5\mathrm{B}_{16}\,(91)$	$2-\frac{1}{91}$

There are some patterns evident in the above table, but they can be generalized by considering more cases. The following table shows the smallest numbers whose reverses are multiples of themselves from bases 3 to 16.

$\frac{\mathrm{rev}(n_b)}{n_b}$	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n	n_{11}	n_{12}
2	1012_3 (32)	–	13_5 (8)	2134_6 (490)	–	25_8 (21)	3256_9 (2400)	–	37_{11} (40)	4378_{12} (7436)
3	–	1023_4 (75)	–	–	15_7 (12)	2156_8 (1134)	–	–	14_{11} (15)	3289_{12} (5577)
4	–	–	1034_5 (144)	–	–	–	17_9 (16)	2178	–	–
5	–	–	–	1045_6 (245)	–	1015_8 (525)	–	–	19_{11} (20)	$219\mathrm{A}_{12}$ (3718)
6	–	–	–	–	1056_7 (384)	–	–	–	–	–
7	–	–	–	–	–	1067_8 (567)	–	–	118_{11} (140)	–
8	–	–	–	–	–	–	1078_9 (800)	–	–	–
9	–	–	–	–	–	–	–	1089	–	–
10	–	–	–	–	–	–	–	–	$109\mathrm{A}_{11}$ (1440)	–
11	–	–	–	–	–	–	–	–	–	$10\mathrm{AB}_{12}$ (1859)

$\frac{\text{rev}(n_b)}{n_b}$	n_{13}	n_{14}	n_{15}	n_{16}
2	–	49_{14} (65)	549A_{15} (17920)	–
3	–	$1\text{A}735_{14}$ (67275)	3B_{15} (56)	43BC_{16} (17340)
4	–	2B_{14} (39)	32BC_{15} (10752)	–
5	18_{13} (21)	–	–	–
6	1B_{13} (24)	21BC_{14} (5850)	–	–
7	–	–	1D_{15} (28)	21DE_{16} (8670)
8	–	–	–	–
9	–	1419B_{14} (49725)	–	–
10	–	–	–	–
11	–	–	102B_{15} (3416)	–
12	10BC_{13} (2352)	–	–	–
13	–	10CD_{14} (2925)	–	–
14	–	–	10DE_{15} (3584)	–
15	–	–	–	10EF_{16} (4335)

A few patterns are clear from this extended table. Given any base b, the smallest number n_b such that $\frac{\text{rev}(n_b)}{n_b} = b - 1$ takes the form:

$$n_b = (b-1)(b+1)^2$$

In contrast, if $b \equiv 0 \pmod 3$, the smallest n_b such that $\frac{\text{rev}(n_b)}{n_b} = 2$ usually takes the form:

$$n_b = \frac{b}{3}(b-1)(b+1)^2$$

When $b \equiv -1 \pmod 3$, the form that doubles is even simpler:

$$n_b = \frac{b+1}{3}(b-1)$$

$b \equiv 1 \pmod 3$ has no number that exactly doubles when reversed, but the smallest n_b for which $|2 - \text{rev}(n_b)| = \frac{1}{n_b}$ is similar to the form that doubles when $b \equiv -1 \pmod 3$:

$$n_b = \frac{b+2}{3}(b-1) + 1$$

These numbers only fail to be the smallest when their digits are not setwise coprime. For example, the smallest n_{11} such that $\frac{\text{rev}(n_{11})}{n_{11}} = 3$ should be 28_{11} per the formula, but it is actually $14_{11} = \frac{28_{11}}{2}$ because both digits are divisible by 2.

These forms are not coincidental, and they can be generalized. As an example, consider the following numbers for base $b = 15$:

$\frac{\text{rev}(n_b)}{n_b} = 2$	$\frac{\text{rev}(n_b)}{n_b} = 3$	$\frac{\text{rev}(n_b)}{n_b} = 4$	$\frac{\text{rev}(n_b)}{n_b} = 7$	$\frac{\text{rev}(n_b)}{n_b} = 14$
549A_{15} (17920) $n_b = 5(b-1)(b+1)^2$	3B_{15} (56) $n_b = 4(b-1)$	32BC_{15} (10752) $n_b = 3(b-1)(b+1)^2$	1D_{15} (28) $n_b = 2(b-1)$	10DE_{15} (3584) $n_b = (b-1)(b+1)^2$

The numbers in the table above take the form $(b-1)$ or $(b-1)(b+1)^2$. At first, they may appear to simply alternate between one form or the other. But when we consider the multiples reached by these numbers when they reverse, we see a stronger pattern:

$$\frac{\text{rev}\left(5(b-1)(b+1)^2\right)}{5(b-1)(b+1)^2} = 2 = \frac{15}{5} - 1$$
$$\frac{\text{rev}\left(3(b-1)(b+1)^2\right)}{3(b-1)(b+1)^2} = 4 = \frac{15}{3} - 1$$
$$\frac{\text{rev}\left((b-1)(b+1)^2\right)}{(b-1)(b+1)^2} = 14 = \frac{15}{1} - 1$$

$$\frac{\text{rev}\,(4(b-1))}{4(b-1)} = 3 = \frac{16}{4} - 1$$
$$\frac{\text{rev}\,(2(b-1))}{2(b-1)} = 7 = \frac{16}{2} - 1$$

The pattern obeys the following rules given a base b and an integer $a > \frac{b}{2}$. If $a|b$:

$$\text{rev}\left(a(b-1)(b+1)^2\right) = \left(\frac{b}{a} - 1\right)\left(a(b-1)(b+1)^2\right)$$

If $a|(b+1)$:

$$\text{rev}(a(b-1)) = \left(\frac{b+1}{a} - 1\right)(a(b-1))$$

We conjecture this is true for all such a and b.

There are, however, other numbers in the chart that don't have an evident underlying pattern. Consider the following four values, which include base 17:

$$\frac{5101_8}{1015_8} = \frac{2625}{525} = 5$$
$$\frac{811_{11}}{118_{11}} = \frac{980}{140} = 7$$
$$\frac{\text{B9141}_{14}}{\text{1419B}_{14}} = \frac{447525}{49725} = 9$$
$$\frac{\text{EC51}_{17}}{\text{15CE}_{17}} = \frac{72336}{6576} = 11$$

The base of each number in this sequences increases by 3 and the multiple each number reaches increases by 2. Further, each number is divisible by $b-1$, and numbers with even bases are divisible by 3. Beyond these facts, whether these numbers posses a common closed form is unknown.

Further, there seem to be oddball cases where no direct multiples are present. The simplest example with $b \leq 16$ is $b = 13$; $\frac{81_{13}}{18_{13}} = 5$, but neither b nor $b+1$ is a multiple of 5, and $18_{13} = 21$ is not a multiple of $b-1$. We conjecture other cases exist, and no closed form for these is known.

Existing research into this topic appears to be limited. Young (1992)[1] created a rooted-tree method to find all n_b that satisfy $\frac{\mathrm{rev}(n_b)}{n_b} = k$ with a fixed b and k, which are referred to as (b, k)-reverse multiples. al-Tahan (2015)[2] focused on base 10 and developed general forms for all decimal k-reverse multiples. I am not aware of research that identifies patterns across bases. I would like to search for more patterns and ideally produce proofs for my findings herein.

[Last updated: 2024-01-19]

[1] Young, Anne Ludington. "k-Reverse Multiples." *The Fibonacci Quarterly* 30.2 (1992): 126-132.
[2] al-Tahan, Madeleine Ali. "$(10, k)$-Reversible Multiples." arXiv:1503.07848 [math.GM]. doi:10.48550/arXiv.1503.07848.