My Tangent Construction

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I was recently asked to help with the following geometry problem:

> Given a circle centered at point A with radius AB and a point C outside circle AB, using a compass and unmarked straightedge, find the lines tangent to circle AB that intersect point C.

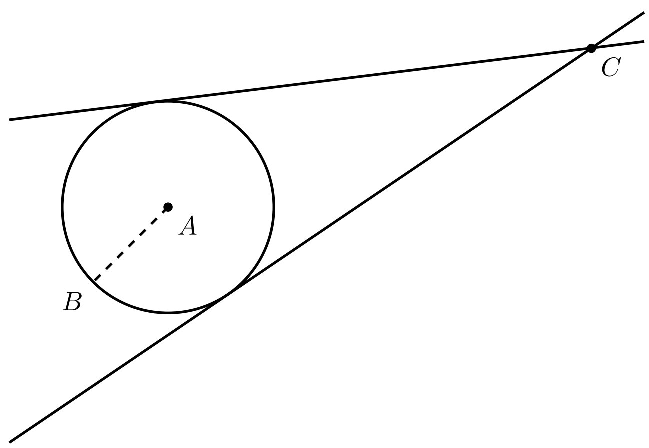

Figure 1: Tangents of $x^2 + y^2 = 4$ intersecting $(8, 3)$

I decided to try to solve it without using any outside references—just my own understanding of geometry. My geometry curriculum in school, while covering some theorems and "pure" constructions, was largely analytical in nature. As a result, I was able to use analytical geometry to determine fairly quickly where the tangent points on the circle would lie, but I found it quite difficult to produce a construction without using angle measurements. It took a few days for me to iron out a solution.

Finally, using similar triangles, I devised the following construction.

We begin by drawing $\overline{AC}$, which intersects circle AB at two points D and E. Draw circle DE (centered at point D with radius DE) and circle ED. These two circles intersect at two points: F and G. Draw $\overline{FG}$. This is a standard construction that creates a line perpendicular to $\overline{AC}$ that passes through point A.

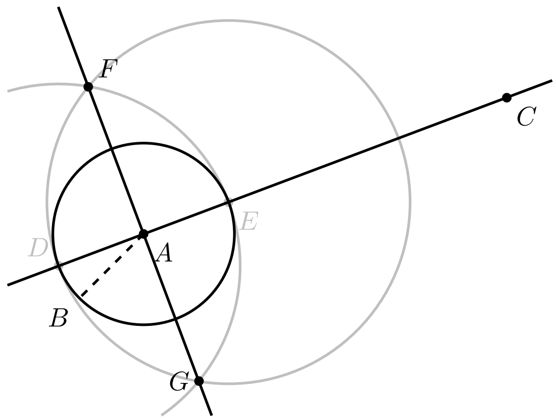

Figure 2: $\overline{FG}$ is perpendicular to $\overline{AC}$ and intersects A

$\overline{FG}$ intersects circle AB at two points. Without loss of generality, call one of these points H. Draw $\overline{CH}$; this intersects circle AB at point I. Draw $\overline{AI}$, which intersects circle AB at point J. Draw $\overline{HJ}$; this intersects $\overline{AC}$ at point K.

This part of the construction creates two right triangles: $\triangle ACH$ and $\triangle HCK$. Note that these triangles are similar.

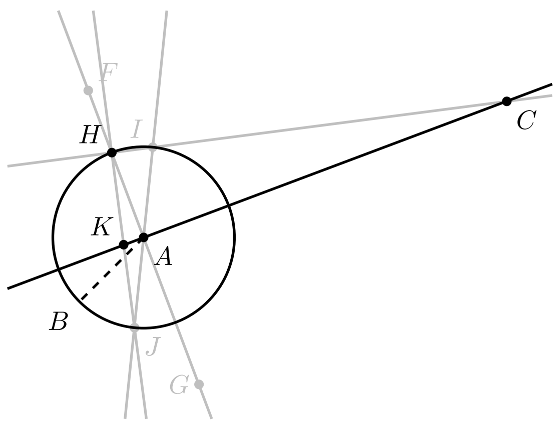

Figure 3: $\triangle ACH$ and $\triangle HCK$ are similar

Draw circle HK; this intersects $\overline{AC}$ at point L. Draw circle LA, which intersects $\overline{AC}$ at point M. Draw circle MA and circle AM. These two circles intersect at points N and O. Draw $\overline{NO}$; this intersects circle AB at points P and Q. $\overline{CP}$ and $\overline{CQ}$ are then the tangents to circle AB that intersect C.

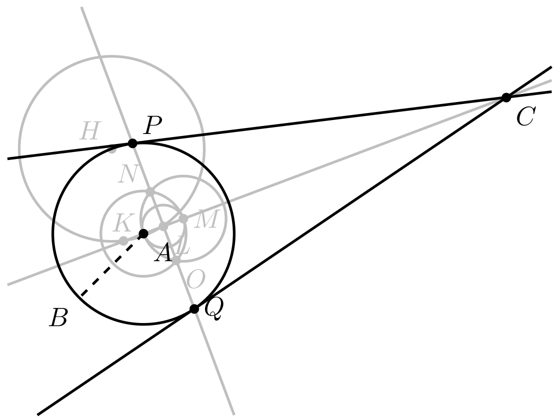

Figure 4: $\overline{CP}$ and $\overline{CQ}$ are tangents

The key to this construction is using similar triangles to construct the point L. To see why this works, we'll work backwards from the tangents.

Fix the center of the circle AB at the origin, set its radius to 1, and suppose point C is located at $(x, 0)$. Then $\triangle ACP$ is a right triangle whose short leg is $AP = 1$ and whose hypotenuse is $AC = x$. Segment LP is an altitude on the hypotenuse of $\triangle ACP$, so $\triangle LPA$ is similar to $\triangle ACP$. The hypotenuse of $\triangle LPA$ is $AP = 1$, which means the length of the short leg AL is $\frac{1}{x}$. We now have a new problem to solve: construct the length $\frac{1}{x}$, from which we can construct the tangents.

Enter the right triangle $\triangle ACH$. This triangle is *not* similar to $\triangle ACP$: its short leg is $AH = 1$ and its long leg is $AC = x$, so its hypotenuse is $CH = \sqrt{x^2+1}$. It is, however, similar to $\triangle HCK$, since $\overline{HK}$ is constructed to be perpendicular to $\overline{CH}$. The long leg of $\triangle HCK$ is $CH = \sqrt{x^2+1}$, so the length of the hypotenuse CK is:

$$\frac{x}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{CK}$$
$$xCK = x^2 + 1$$
$$CK = x + \frac{1}{x}$$

Because the length of AC is x, the length of AK is $\frac{1}{x}$, our desired length.

The only steps left are to mirror point K on the other side of $\overline{AH}$, giving us L, and to construct a perpendicular line to $\overline{AC}$ that intersects L. The full construction takes 12 steps to find points P and Q.

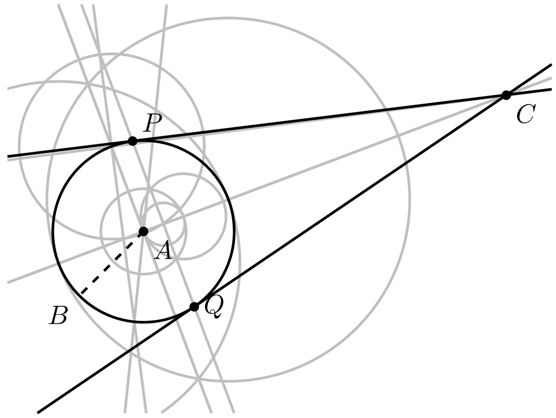

Figure 5: The full construction has many steps

After I finished my construction, I looked up the problem online to see if there was a "standard" construction—and it didn't take long to find one. The following construction takes only five steps.

Begin by drawing $\overline{AC}$. Draw circle AC and circle CA; these intersect at points D and E. Draw $\overline{DE}$, which intersects $\overline{AC}$ at point F. Draw circle FA; this intersects circle AB at points G and H. $\overline{CG}$ and $\overline{CH}$ are the tangents to circle AB that intersect C.

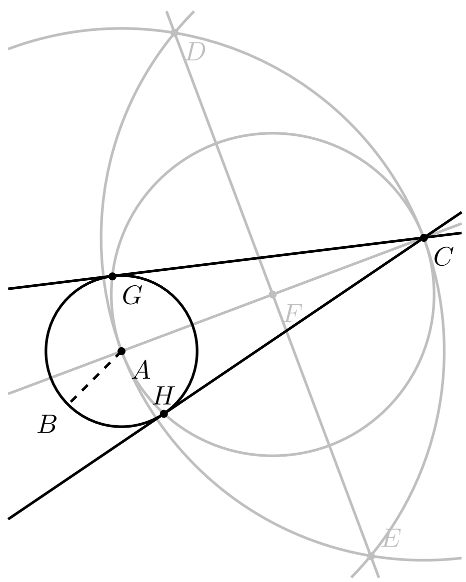

Figure 6: The standard construction takes five steps

Unlike my construction, the standard construction uses purely geometric concepts. Specifically, point F is the midpoint of AC, constructed using the perpendicular bisector of AC. As noted, A, C and the tangent point G form a right triangle, of which AC is the hypotenuse. An important property of a right triangles is that the center of its circumcircle is located on the midpoint of its hypotenuse; the standard construction uses this fact to determine the location of G (and H) using F directly.

To me, this is a good example of different methods of problem solving. A weakness of mine is that I tend to take a "one tool fixes all" approach to problem solving: if one method works, even if it's inefficient, well, at least it works. In this case, I'm well-versed in analytical geometry, and I was able to use it to find a solution, clunky as it is. Were I better versed in pure geometry and the properties of polygons—a seemingly-unrelated concept—I might have been able to find this construction on my own.

Many real-world problems require optimization and efficiency, sometimes even at the cost of correctness in corner cases. In order to find such optimizations, it's often necessary to attack a problem from many different angles. This is the very definition of thinking outside the box.

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