# Another Fractional Problem

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I first came across this problem on Brilliant many years ago. We are asked to find all real solutions $1 \le x \le 6$ to the equation

$$\{x\} = \left\{\frac{1}{x}\right\}$$

$\{x\}$ is called the "fractional part" of $x$ and is defined as $\{x\} = x - \lfloor x \rfloor$, $\lfloor x \rfloor$ denoting the floor function and defined as the largest integer $m$ such that $m \le x$. In other words, $\{x\}$ is the non-integral part of a given number $x$.

Because $\{x\} = \left\{\frac{1}{x}\right\}$, $\{x\} - \left\{\frac{1}{x}\right\} = 0$. Of course, this equation only refers to the fractional parts of $x$ and $\frac{1}{x}$; the actual result of $x - \frac{1}{x}$ might not be 0. However, our equation does tell us that the expression $x - \frac{1}{x}$ has no fractional part and is thus integral.

$$x - \frac{1}{x} = k, \qquad k \in \mathbb{Z}$$

Because $x > 0$, this translates to the quadratic equation

$$x^2 - kx - 1 = 0$$

Putting the equation in terms of $k$ yields

$$x = \frac{k \pm \sqrt{k^2 + 4}}{2}$$

$k$ can be any integer, but $x$—which is a root of the equation–must be between 1 and 6. How can we narrow down the possible values of $k$? Fortunately, we can drastically reduce our potential solution set by comparing the problem's boundary conditions to some of our equation's properties.

We first observe that $\sqrt{k^2+4} > k$, so $k - \sqrt{k^2+4} < 0$, which violates our lower bound for $x$. This means we only need to check one possible value for each k.
Further, $\sqrt{k^2+4} < |k| + 2$ for any $|k| > 0$, so if $k < 0$:

$$\frac{k + \sqrt{k^2+4}}{2} < \frac{-|k| + |k| + 2}{2} = \frac{2}{2} = 1$$

This also violates our lower bound for x, so k cannot be negative.
Finally, for $k > 0$:

$$\frac{k + \sqrt{k^2+4}}{2} > \frac{k + k}{2} = k$$

Specifically, if $k \ge 6$, we violate our upper bound for $x$. We thus only need to consider

$$k = 0, 1, 2, 3, 4, 5$$

These values of $k$ yield the following values for $x$:

$$\begin{aligned}
k &= 0, \quad x = 1 \\
k &= 1, \quad x = \frac{1+\sqrt{5}}{2} \\
k &= 2, \quad x = 1+\sqrt{2} \\
k &= 3, \quad x = \frac{3+\sqrt{13}}{2} \\
k &= 4, \quad x = 2+\sqrt{5} \\
k &= 5, \quad x = \frac{5+\sqrt{29}}{2}
\end{aligned}$$

All of these values are between 1 and 6, so they are all valid solutions.

This problem is in a similar vein to the previous fractional problem: using definitions to rewrite the original equation, then using boundary conditions to reduce the number of possible solutions. In that problem, however, the initial solution was open-ended, and only implied conditions allowed us to find the right range to impose on the solution. Here we use explicit conditions to facilitate the search for potential solutions, narrowing a potential infinite search to just six possible candidates. This use case is a classic example of optimization.

I deal with optimization questions quite often in IT. Computers can only perform so many operations per second; therefore it's important to try to find results in as few operations as possible. Pure mathematics does not suffer from this limitation—a true theorem is true, regardless of how much computational power is needed to prove it in practice. Problems like this give students a chance to use boundary conditions not to merely find a correct solution, but to find it quickly and easily. That's an important skill to emphasize in any applied mathematical field.

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