Integrating the Standard Normal Distribution

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2021-11-18

One of the most prevalent probability distributions is the standard normal distribution, given by the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Like all probability distributions, the area under this function is 1, but the integral of the base function e^{-x^2} cannot be expressed in terms of elementary functions. Instead, these kinds of integrals are expressed in terms of special functions called "error functions". How, then, can we show that the area under the curve is unit?

I found the following solution online many years ago while I was still a student, but I didn't bother to go through the logic of each step. This note goes through the justification for the solution method and what can be learned from it.

We wish to determine the value of the definite integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

$\frac{1}{\sqrt{2\pi}}$ is a constant factor, so it can be separated from the integrand.
Also, because x^2 is an even function, e^{-x^2} is an even function. We can thus consider the integral only from 0 to ∞.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{-x^2}{2}} dx$$

To aid in evaluating the integral, we will introduce another term into the integrand. We do this by mutiplying the integral by itself and taking its square root.

$$\frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{-x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \sqrt{\int_{0}^{\infty} e^{\frac{-x^2}{2}} dx \int_{0}^{\infty} e^{\frac{-x^2}{2}} dx}$$

Definite integrals are not affected by choice of variable, so we substitute x with y in the second integral. This allows us to join the two integrals into one double integral.

$$\begin{aligned} \frac{2}{\sqrt{2\pi}} \sqrt{\int_{0}^{\infty} e^{\frac{-x^2}{2}} dx \int_{0}^{\infty} e^{\frac{-x^2}{2}} dx} &= \frac{2}{\sqrt{2\pi}} \sqrt{\int_{0}^{\infty} e^{\frac{-x^2}{2}} dx \int_{0}^{\infty} e^{\frac{-y^2}{2}} dy} \\ &= \frac{2}{\sqrt{2\pi}} \sqrt{\int_{0}^{\infty} \int_{0}^{\infty} e^{\frac{-(x^2+y^2)}{2}} dx dy} \end{aligned}$$

We now perform another change of variable:

$$y = sx, \qquad dy = x ds$$

Substituting:

$$\begin{aligned}
\frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty\int_0^\infty e^{\frac{-(x^2+y^2)}{2}}dxdy} &= \frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty\int_0^\infty xe^{\frac{-(x^2+s^2x^2)}{2}}dxds} \\
&= \frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty\int_0^\infty xe^{\frac{-x^2}{2}(1+s^2)}dxds}
\end{aligned}$$

Having x in front of the exponent allows us to integrate with respect to x and produce elementary functions.

$$\begin{aligned}
\frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty\int_0^\infty xe^{\frac{-x^2}{2}(1+s^2)}dxds} &= \frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty \left[\left(\frac{-1}{1+s^2}\right)e^{\frac{-x^2}{2}(1+s^2)}\right]_0^\infty ds} \\
&= \frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty \left(\left(\frac{-1}{1+s^2}\right)(e^{-\infty}-e^0)\right)ds} \\
&= \frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty \frac{1}{1+s^2}ds}
\end{aligned}$$

The remaining integrand is the derivative of the arctangent function. After evaluating the integral, the final arithmetic is simple.

$$\begin{aligned}
\frac{2}{\sqrt{2\pi}}\sqrt{\int_0^\infty \frac{1}{1+s^2}ds} &= \frac{2}{\sqrt{2\pi}}\sqrt{[\arctan s]_0^\infty} \\
&= \frac{2}{\sqrt{2\pi}}\sqrt{\arctan\infty-\arctan 0} \\
&= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} \\
&= \frac{2\sqrt{\pi}}{\sqrt{2}\sqrt{2}\sqrt{\pi}} \\
&= 1
\end{aligned}$$

The core of this solution is to create a double integral and apply a clever change of variable to modify the integrand. Often the difference between an elementary integral and an nonelementary integral is no more than a leading factor—in this case the presence of x in front of the term e^{-x^2}. Introducing a second integral and converting to a double integral allowed us to put in the factor we needed.

I tend to struggle with integration problems. I seem to lack the creativity needed to find an extra term or change of variable needed to simplify the problem. It might be that I simply don't remember my basic integrals well enough to realize what might be useful. At any rate, solutions like this fascinate me.

[Last updated: 2021-11-19]