Analyzing a Trigonometric Cubic Formula

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A short and relatively-obscure page on Wikibooks[1] provides the following intriguing trigonometric method to find the roots of certain cubic equations. (The form presented below is abridged.)

We consider a monic cubic equation in one variable $x^3 + ax^2 + bx + c = 0$, where a, b, c are real and the discriminant $\Delta \geq 0$ (all three roots are real).[2] Let

$$p = \frac{3b - a^2}{9}, \quad q = \frac{9ab - 2a^3 - 27c}{54},$$

$$\cos\theta = \frac{q}{\sqrt{-p^3}}$$

The three roots of the polynomial are then

$$x_n = 2\sqrt{-p}\cos\left(\frac{\theta + 2\pi n}{3}\right) - \frac{a}{3}, \quad n = 0, 1, 2$$

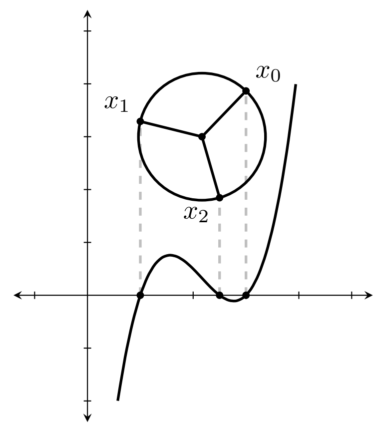

Figure 1: $x^3 - \frac{13}{2}x^2 + 13x - \frac{15}{2} = 0$

The Wikibooks page does not cite references for this method, and I haven't seen it in other sources, so I'm not sure where it came from. My goal with this note is to try to figure out how this method works.

A few facts immediately stand out to me.

- The x coordinate for the center of the generating circle is $a/3$, the arithmetic mean of the three roots. Because it depends only on a, it does not move when b or c change.
- The final equation involves the term $\cos\frac{\theta}{3}$. Expanding this out in terms of $\cos\theta$ creates a cubic expression of its own.

The second fact seems like a good place to start. The cosine of a triple angle expands as follows:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

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[1] https://en.wikibooks.org/wiki/Trigonometry/The_solution_of_cubic_equations

[2] $\Delta = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$

This expression does not include a square term. Such expressions are known as depressed cubics. We can depress our original equation using the substitution $x = t - \frac{a}{3}$.

$$\begin{aligned}
x^3 + ax^2 + bx + c &= (t - \frac{a}{3})^3 + a(t - \frac{a}{3})^2 + b(t - \frac{a}{3}) + c \\
&= t^3 - at^2 + \frac{a^2}{3}t - \frac{a^3}{27} + at^2 - \frac{2a^2}{3}t + \frac{a^3}{9} + bt - \frac{ab}{3} + c \\
&= t^3 + ut + v, \quad u = \frac{3b - a^2}{3}, \quad v = \frac{2a^3 - 9ab + 27c}{27}
\end{aligned}$$

It is immediately apparent that $u = 3p$ and $v = -2q$. This approach seems promising.

The first fact tells us that when considering a depressed cubic, the center of the generating circle for the roots is at $x = 0$. This means all three roots of the depressed cubic can be expressed as $r\cos\phi$ for some angle ϕ and radius r. Set $t = r\cos\phi$:

$$\begin{aligned}
t^3 + ut + v &= r^3\cos^3\phi + \left(\frac{3b - a^2}{3}\right) r\cos\phi + \frac{2a^3 - 9ab + 27c}{27} \\
&= r^2\left(r\cos^3\phi - \frac{3r}{4}\cos\phi\right) + \left(\frac{3b - a^2}{3} + \frac{3r^2}{4}\right) r\cos\phi + \frac{2a^3 - 9ab + 27c}{27} \\
&= \frac{r^3}{4}\cos 3\phi + \left(\frac{9r^2 + 12b - 4a^2}{12}\right) r\cos\phi + \frac{2a^3 - 9ab + 27c}{27}
\end{aligned}$$

We now have mixed cosine terms in our equation: $\cos\phi$ and $\cos 3\phi$. However, we can remove the $\cos\phi$ term by choosing r such that the coefficient vanishes.

$$\frac{9r^2 + 12b - 4a^2}{12} = 0$$

$$r = \frac{2}{3}\sqrt{a^2 - 3b}$$

Note that $r = 2\sqrt{-p}$. Substituting this value for r yields

$$\sqrt{\left(\frac{a^2 - 3b}{9}\right)^3}\cos 3\phi + \frac{2a^3 - 9ab + 27c}{54} = 0$$

Since

$$\sqrt{\left(\frac{a^2 - 3b}{9}\right)^3} = \sqrt{-p^3} \quad \text{and} \quad \frac{2a^3 - 9ab + 27c}{54} = -q$$

our final result is

$$\begin{aligned}
\sqrt{-p^3}\cos 3\phi - q &= 0 \\
\cos 3\phi &= \frac{q}{\sqrt{-p^3}} \\
&= \cos\theta
\end{aligned}$$

which is the exact result we desire. Since $\cos\phi = \cos(\phi + 2\pi) = \cos(\phi + 4\pi)$, we find three valid solutions in terms of θ: $\cos\left(\frac{\theta}{3}\right)$, $\cos\left(\frac{\theta + 2\pi}{3}\right)$ and $\cos\left(\frac{\theta + 4\pi}{3}\right)$.

In terms of θ, the three solutions of the depressed cubic are thus

$$t_n = 2\sqrt{-p}\cos\frac{\theta + 2\pi n}{3}, \quad n = 0, 1, 2$$

After substituting x for t, the three solutions of the original cubic are

$$x_n = 2\sqrt{-p}\cos\frac{\theta + 2\pi n}{3} - \frac{a}{3}, \quad n = 0, 1, 2$$

as the above solution claims.

While the mechanics of this cubic formula involve nothing more than depressing a cubic and substituting a trigonometric value, I still find it fascinating. Cubic formulas involving surds exist, but they are quite complicated and messy. This formula, by contrast, is rather elegant. One more reason not to underestimate the power of trigonometry.

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